

THE RIGHT CLASSIFICATION OF UNIVARIATE POWER SERIES IN POSITIVE CHARACTERISTIC

NGUYEN HONG DUC

ABSTRACT. While the classification of univariate power series up to coordinate change is trivial in characteristic 0, this classification is very different in positive characteristic. In this note we give a complete classification of univariate power series $f \in K[[x]]$, where K is an algebraically closed field of characteristic $p > 0$ by explicit normal forms. Moreover we show that the right modality of f is equal to the integer part of μ/p , where μ is the Milnor number of f . As a consequence we prove in this case that the modality is equal to the inner modality, which is the dimension of the μ -constant stratum in an algebraic representative of the semiuniversal deformation with trivial section.

1. INTRODUCTION

In [Arn72] V.I. Arnol'd introduced the “modality”, or the number of moduli, for real and complex singularities and he classified singularities with modality smaller than 2. In order to generalize the notion of modality to algebraic setting, the author and Greuel in [GN12] introduced the modality for algebraic actions and applied it to high jet spaces.

Let the algebraic group G act on the variety X . Then there exists a *Rosenlicht stratification* $\{(X_i, p_i), i = 1, \dots, s\}$ of X w.r.t. G , i.e. the X_i is a locally closed G -invariant subset of X , $X = \cup_{i=1}^s X_i$ and the $p_i : X_i \rightarrow X_i/G$ a geometric quotient. For each open subset $U \subset X$ we define

$$G\text{-mod}(U) := \max_{1 \leq i \leq s} \{\dim(p_i(U \cap X_i))\},$$

and for $x \in X$ we call

$$G\text{-mod}(x) := \min\{G\text{-mod}(U) \mid U \text{ a neighbourhood of } x\}$$

the *G-modality* of x .

Let K be an algebraically closed field of characteristic $p \geq 0$, let $K[[\mathbf{x}]] = K[[x_1, \dots, x_n]]$ the formal power series ring and let the right group, $\mathcal{R} := \text{Aut}(K[[\mathbf{x}]])$, act on $K[[\mathbf{x}]]$ by $(\Phi, f) \mapsto \Phi(f)$. Two elements $f, g \in K[[\mathbf{x}]]$ are called *right equivalent*, $f \sim_r g$, iff they belong to the same \mathcal{R} -orbit, or equivalently, there exists a coordinate change $\Phi \in \text{Aut}(K[[\mathbf{x}]])$ such that $g = \Phi(f)$.

Recall that for $f \in \langle \mathbf{x} \rangle \subset K[[\mathbf{x}]]$, $\mu(f) := \dim K[[\mathbf{x}]]/\langle f_{x_1}, \dots, f_{x_n} \rangle$ denotes the *Milnor number* of f and that f is isolated if $\mu(f) < \infty$. By [GN12, Prop. 2.15] every $k \geq 2\mu(f)$ is *sufficiently large* for the isolated singularity $f \in K[[\mathbf{x}]]$ w.r.t. \mathcal{R} , i.e. there exists a neighbourhood U of the k -jet of f , denoted by $j^k f$, in the k -th jet space $J_k := \langle \mathbf{x} \rangle / \langle \mathbf{x} \rangle^{k+1}$, s.t. every $g \in K[[\mathbf{x}]]$ with $j^k g \in U$ is *right k-determined*. This means that each $h \in K[[\mathbf{x}]]$ s.t. $j^k h = j^k g$, is right equivalent to g . The

right modality of f , $\mathcal{R}\text{-mod}(f)$, is defined to be the \mathcal{R}_k -modality of $j^k f$ in J_k with k sufficiently large for f w.r.t. \mathcal{R} and \mathcal{R}_k the k -jet of \mathcal{R} .

In Section 2, we give a normal form for univariate power series f w.r.t. right equivalence (Prop. 2.8). We show in Section 3 that the right modality of f is equal to the integer part of $\mu(f)/p$ (Thm. 3.1). As a consequence we prove that the right modality is equal to the dimension of the μ -constant stratum in an algebraic representative of the semiuniversal deformation with trivial section (Cor. 3.5).

The result of this article is part of my thesis [Ng12] under the supervision of Professor Gert-Martin Greuel at the Technische Universität Kaiserslautern. I am grateful to him for many valuable suggestions. This work is supported by DAAD (Germany) and NAFOSTED (Vietnam).

2. NORMAL FORMS OF UNIVARIATE POWER SERIES

Let $f = \sum_{n \geq 0} c_n x^n \in K[[x]]$ be a univariate power series, let $\text{supp}(f) := \{n \geq 0 \mid c_n \neq 0\}$ be the support of f and $m(f) := \min\{n \mid n \in \text{supp}(f)\}$ the multiplicity of f . If $\text{char}(K) = 0$ and if $\varphi(x) = a_1 x + a_2 x^2 + \dots, a_1 \neq 0$, is a coordinate change, then the coefficients a_i of φ can be determined inductively from the equation $f(x) = c_0 + (\varphi(x))^{m(g)}$ with $g(x) := f - c_0$. Hence f is right equivalent to $c_0 + x^{m(g)}$.

In the following we investigate $f \in K[[x]]$ with $\text{char}(K) = p > 0$. The aim of this section is to give a normal form of f . It turns out that it depends in a complicated way from the divisibility relation between p and the support of f . To describe this relation we make the following definition, where later on Δ will be $\text{supp}(f)$.

Definition 2.1. For each subset $\Delta \subset \mathbb{N} \setminus \{0\}$, we define

- (a) $m(\Delta) := \min\{n \mid n \in \Delta\}$
- (b) $e(\Delta) := \min\{e(n) \mid n \in \Delta\}$, where $e(n) := \max\{i \mid p^i \text{ divides } n\}$.
- (c) $q(\Delta) := \min\{n \mid e(n) = e(\Delta), n \in \Delta\}$.
- (d) $k(\Delta) := 1$ if $m(\Delta) = q(\Delta)$, otherwise, $k(\Delta) := \max\{k_\Delta(n) \mid m(\Delta) \leq n < q(\Delta), n \in \Delta\}$,
where

$$k_\Delta(n) := \left\lceil \frac{q(\Delta) - n}{p^{e(n)} - p^{e(\Delta)}} \right\rceil \text{ denotes the ceiling of } \frac{q(\Delta) - n}{p^{e(n)} - p^{e(\Delta)}}.$$

- (e) $\bar{d}(\Delta) := 2q(\Delta) - m(\Delta)$ and $d(\Delta) := q(\Delta) + p^{e(\Delta)}(k(\Delta) - 1)$.
- (f) $\bar{\Lambda}(\Delta) := \{n \mid m(\Delta) < n \leq \bar{d}(\Delta), e(\Delta) < e(n)\} \cup \{q(\Delta)\}$.
- (g) If $e(m(\Delta)) > e(\Delta)$ we define
 $\Delta_0 := \{n \in \Delta \mid n < q(\Delta)\}$, $e_0 := e(\Delta_0)$ and $q_0 := q(\Delta_0)$.
 $\Lambda_0(\Delta) := \bar{\Lambda}(\Delta_0) \cap \mathbb{N}_{< q(\Delta)}$.
- (h) For each subset Δ with $e(\Delta) = 0$ we denote
 $\Lambda_1(\Delta) := \{n \in \mathbb{N} \mid q(\Delta) \leq n \leq d(\Delta)\}$
 $\Lambda'_1(\Delta) := \{n \in \mathbb{N} \mid q(\Delta) \leq n \leq d(\Delta), e(n) \neq 1\}$
 $\Lambda''_1(\Delta) := \{n \in \mathbb{N} \mid q(\Delta) \leq n \leq d(\Delta), e(n) = 0\}$

and define the set $\Lambda(\Delta)$ as follows:

Case 0: If $e(m(\Delta)) = 0$ then $\Lambda(\Delta) := \emptyset$.

Case 1: If $e(m(\Delta)) > 0$ and if $e(m(\Delta)) = e_0$, then

$$\Lambda(\Delta) := \{n \in \mathbb{N} \mid q(\Delta) \leq n \leq d(\Delta), e(n) < e(m(\Delta))\}.$$

(Note that in this case one has $k(\Delta) = k_\Delta(m(\Delta))$).

Case 2: If $e(m(\Delta)) > e_0 > 1$ then $\Lambda(\Delta) := \Lambda_0(\Delta) \cup \Lambda_1(\Delta)$.

Case 3: If $e(m(\Delta)) > e_0 = 1$ and if $k(\Delta) > k(q_0)$, then $\Lambda(\Delta) := \Lambda_0(\Delta) \cup \Lambda_1(\Delta)$.

Case 4: If $e(m(\Delta)) > e_0 = 1$ and $k(\Delta) = k_\Delta(q_0)$ and if $k(\Delta_0) \geq \lfloor \frac{q-q_0}{p} \rfloor$ then

$$\Lambda(\Delta) := \Lambda_0(\Delta) \cup \Lambda'_1(\Delta).$$

Case 5: If $e(m(\Delta)) > e_0 = 1$ and $k(\Delta) = k_\Delta(q_0)$ and if $k(\Delta_0) < \lfloor \frac{q-q_0}{p} \rfloor$ then

$$\Lambda(\Delta) := \Lambda_0(\Delta) \cup \Lambda''_1(\Delta).$$

Remark 2.2. The following facts (a)-(d) are immediate consequences of the definition. Property (e) follows from elementary calculations.

(a) If p does not divide $m(\Delta)$, then

1. $e(\Delta) = e(m(\Delta)) = 0$ and $q(\Delta) = m(\Delta)$.
2. $k(\Delta) = 0$ and $d(\Delta) = m(\Delta)$.

(b) If $e(m(\Delta)) = e(\Delta)$, then

1. $q(\Delta) = m(\Delta)$.
2. $k(\Delta) = 0$ and $d(\Delta) = m(\Delta)$.

(c) If $q(\Delta) = q(\Delta') =: q$ and $\Delta \cap \mathbb{N}_{<q} = \Delta' \cap \mathbb{N}_{<q}$, then $\Lambda(\Delta) = \Lambda(\Delta')$.

(d) $d(\Delta) \leq \bar{d}(\Delta)$.

(e) If $e(\Delta) = 0$ and $k(\Delta) = k_\Delta(n)$, then

$$k(\Delta) - \frac{n}{p^{e(n)}} + 1 = \lfloor \frac{d(\Delta)}{p^{e(n)}} \rfloor.$$

The following proposition is the first key step in the classification.

Proposition 2.3. *Let $\Delta \subset \mathbb{N} \setminus \{0\}$ such that $e(\Delta) = 0$. Then*

$$\sharp \Lambda(\Delta) \leq \lfloor \frac{q(\Delta)}{p} \rfloor, \text{ the interger part of } \frac{q(\Delta)}{p}.$$

Equality holds if $m(\Delta) = p$.

Proof. We first proof the second statement. Assume that $m(\Delta) = p$. Then

$$\Lambda(\Delta) = \{n \mid q(\Delta) \leq n \leq d(\Delta), e(n) = 0\}$$

and hence

$$\sharp \Lambda(\Delta) = k(\Delta) - (\lfloor \frac{d(\Delta)}{p} \rfloor - \lfloor \frac{q(\Delta)}{p} \rfloor) = \lfloor \frac{q(\Delta)}{p} \rfloor$$

since $k(\Delta) = \lfloor \frac{d}{p} \rfloor$ due to Remark 2.2(e).

In order to prove the first statement we may restrict to the case that $e(m(\Delta)) > 0$ since the case $e(m(\Delta)) = 0$ is trivial. We set

$$m := m(\Delta), q := q(\Delta), d := d(\Delta), k := k(\Delta), q_0 := q(\Delta_0), e_0 := e(\Delta_0).$$

Case 1a: $e(m) = e_0 \geq 2$.

Then $k = k_\Delta(m) \leq \lceil \frac{q-m}{p^2-1} \rceil \leq \lfloor \frac{q}{p} \rfloor$. This implies that

$$\sharp\Lambda(\Delta) \leq k \leq \lfloor \frac{q}{p} \rfloor.$$

Case 1b: $e(m) = e_0 = 1$.

Then $k = k_\Delta(m) = \lceil \frac{q-m}{p-1} \rceil \leq \lceil \frac{q-p}{p-1} \rceil = k(\Delta_1)$ with $\Delta_1 := \{p, q\}$. It follows that

$$\Lambda(\Delta) \subset \Lambda(\Delta_1) = \{n \mid q \leq n \leq d(\Delta_1), e(n) = 0\}$$

and hence

$$\sharp\Lambda(\Delta) \leq \sharp\Lambda(\Delta_1) = \lfloor \frac{q}{p} \rfloor$$

by the second statement of the proposition.

Case 2: $e(m) > e_0 \geq 1$.

Then $k = k_\Delta(n)$ for some $n \in \Delta$ and $e(n) \geq 2$. It yields that

$$k = k_\Delta(n) = \lceil \frac{q-n}{p^{e(n)}-1} \rceil \leq \lceil \frac{q-n}{p^2-1} \rceil < \lfloor \frac{q}{p^2-1} \rfloor.$$

Since $\Lambda_0(\Delta) = \bar{\Lambda}(\Delta_0) = (\{n \mid m < n \leq \bar{d}(\Delta_0), e_0 < e(n)\} \cap \mathbb{N}_{< q}) \cup \{q_0\}$,

$$\sharp\Lambda_0(\Delta) \leq \lfloor \frac{q}{p^{e_0+1}} \rfloor - \lfloor \frac{m}{p^{e_0+1}} \rfloor + 1 \leq \lfloor \frac{q}{p^3} \rfloor.$$

This implies that

$$\sharp\Lambda(\Delta) = \sharp\Lambda_0(\Delta) + \sharp\Lambda_1(\Delta) < \lfloor \frac{q}{p^2-1} \rfloor + \lfloor \frac{q}{p^3} \rfloor < \lfloor \frac{q}{p} \rfloor.$$

Case 3: $e(m) > e_0 = 1$ and $k > k_\Delta(q_0)$.

Using the same argument as in Case 2 we get

$$\sharp\Lambda(\Delta) \geq \lfloor \frac{q}{p} \rfloor.$$

Case 4: $e(m) > e_0 = 1$ and $k = k_\Delta(q_0)$ and $k(\Delta_0) \geq \lfloor \frac{q-q_0}{p} \rfloor$.

Then

$$\sharp\Lambda_0(\Delta) \leq \lfloor \frac{q}{p^2} \rfloor - \lfloor \frac{m}{p^2} \rfloor \leq \lfloor \frac{q}{p^2} \rfloor - 1$$

and

$$\begin{aligned} \sharp\Lambda'_1(\Delta) &= k - (\lfloor \frac{d}{p} \rfloor - \lfloor \frac{q}{p} \rfloor) + (\lfloor \frac{d}{p^2} \rfloor - \lfloor \frac{q}{p^2} \rfloor) \\ &= \lfloor \frac{q}{p} \rfloor - \frac{q_0}{p} + \lfloor \frac{d}{p^2} \rfloor - \lfloor \frac{q}{p^2} \rfloor + 1 \end{aligned}$$

since $k - \lfloor \frac{d}{p} \rfloor = -\frac{q_0}{p} + 1$ according to Remark 2.2(e). Hence

$$\sharp\Lambda(\Delta) = \sharp\Lambda_0(\Delta) + \sharp\Lambda'_1(\Delta) \leq \lfloor \frac{q}{p} \rfloor - \frac{q_0}{p} + \lfloor \frac{d}{p^2} \rfloor.$$

Moreover since $k(\Delta_0) \geq \lfloor \frac{q-q_0}{p} \rfloor$,

$$\lceil \frac{q_0 - n}{p^{e(n)} - p} \rceil \geq \lfloor \frac{q - q_0}{p} \rfloor$$

for some $n \in \Delta_0$ with $e(n) > 1$. It follows easily that $pq_0 \geq d$ and hence

$$\sharp\Lambda(\Delta) \leq \lfloor \frac{q}{p} \rfloor - \frac{q_0}{p} + \lfloor \frac{d}{p^2} \rfloor \leq \lfloor \frac{q}{p} \rfloor.$$

Case 5: $e(m) > e_0 = 1$ and $k = k_\Delta(q_0)$ and $k(\Delta_0) < \lfloor \frac{q-q_0}{p} \rfloor$.

Then

$$\sharp\Lambda_0(\Delta) \leq \lfloor \frac{2q_0 - m}{p^2} \rfloor - \lfloor \frac{m}{p^2} \rfloor \leq \lfloor \frac{2q_0}{p^2} \rfloor - 2.$$

and

$$\sharp\Lambda_1''(\Delta) = k - (\lfloor \frac{d}{p} \rfloor - \lfloor \frac{q}{p} \rfloor) = \lfloor \frac{q}{p} \rfloor - \frac{q_0}{p} + 1.$$

Hence

$$\sharp\Lambda(\Delta) = \sharp\Lambda_0(\Delta) + \sharp\Lambda_1''(\Delta) < \lfloor \frac{q}{p} \rfloor - \frac{q_0}{p} + \lfloor \frac{2q_0}{p^2} \rfloor \leq \lfloor \frac{q}{p} \rfloor.$$

□

Note that if $m(f) = 0$ then $m(f - f(0)) > 0$. Applying the results from $m(f) > 0$ to $f - f(0)$ we obtain that $f \sim_r f(0) + g$, where g is a normal form of $f - f(0)$ (cf. Theorem 2.8). From now we assume that $m(f) > 0$. We denote

- (a) $e(f) := e(\text{supp}(f))$, $q(f) := q(\text{supp}(f))$, $k(f) := k(\text{supp}(f))$, $\bar{d}(f) := \bar{d}(\text{supp}(f))$, $d(f) := d(\text{supp}(f))$ and $\bar{\Lambda}(f) := \bar{\Lambda}(\text{supp}(f))$.
- (b) $\Lambda(f) := \{n \cdot p^{e(f)} \mid n \in \Lambda(\text{supp}(\bar{f}))\}$ with \bar{f} defined as in Remark 2.4(b) below.

Note that if $e(f) = 0$ then $\Lambda(f) = \Lambda(\text{supp}(f))$.

Remark 2.4. (a) The above numbers m, e, q, k, \bar{d} and d are invariant w.r.t. right equivalence.

(b) Let $f = \sum_{n \geq 1} c_n x^n \in K[[x]]$ and let

$$\bar{f}(x) = \sum_{n \geq m(f)} c_n x^{n/p^{e(f)}}.$$

Then $\bar{f} \in K[[x]]$, $f(x) = \bar{f}(x^{p^{e(f)}})$ and $e(\bar{f}) = 0$. Moreover, $k(f) = k(\bar{f})$ and if $\zeta(f)$ denotes one of $m(f), e(f), q(f), \bar{d}(f), d(f)$ then

$$\zeta(f) = p^{e(f)} \zeta(\bar{f}).$$

- (c) Note that $\mu(f) < \infty$ if and only if $e(f) = 0$ and then $q(f) = \mu(f) + 1$ and $\bar{d}(f) = 2\mu(f) - m + 2$. Moreover, f is then right $\bar{d}(f)$ -determined by [BGM12, Thm. 2.1]. In Proposition 2.7 below we give a better bound for determinacy of a univariate power series.

Lemma 2.5. If $e(m(f)) = e(f)$ then $f \sim_r x^{m(f)}$.

Proof. By Remark 2.4, there exists $\bar{f} \in K[[x]]$ such that $f(x) = \bar{f}(x^{p^{e(f)}})$ and $e(\bar{f}) = 0$. This implies that $\mu(\bar{f}) = q(\bar{f})$ and then $\mu(\bar{f}) = m(\bar{f}) - 1$ since $e(m(f)) = e(f)$. It follows from [BGM12, Thm. 2.1] that \bar{f} is right $(m(\bar{f}) + 1)$ -determined and hence

$$\bar{f} \sim_r c_m x^{m(\bar{f})} \sim_r x^{m(\bar{f})}.$$

In fact, in this case an inductive proof as in the case of characteristic 0 works. \square

Lemma 2.6. *A univariate power series $f \in K[[x]]$ is right equivalent to*

$$x^{m(f)} + \sum_{n \in \bar{\Lambda}(f)} \lambda_n x^n,$$

for suitable $\lambda_n \in K$.

Proof. It is sufficient to prove the lemma for the case that $e(f) = 0$ by Remark 2.4. We decompose $f = f_0 + f_1$ with

$$f_0 := \sum_{e(i)=0} c_i x^i \text{ and } f_1 := \sum_{e(n)>0} c_n x^n.$$

Then $m(f_0) = q(f)$ and $e(m(f_0)) = e(f_0) = 0$ and hence $f_0 \sim_r x^{q(f)}$ by Lemma 2.5. That is, $\varphi(f_0) = x^{q(f)}$ for some coordinate change $\varphi \in \text{Aut}(K[[x]])$. It yields that

$$g := \varphi(f) = \varphi(f_0) + \varphi(f_1) = x^{q(f)} + \varphi(f_1).$$

This implies that $f \sim_r g \sim_r j^{\bar{d}(g)}(g)$ due to Remark 2.4(c). It is easy to see that $j^{\bar{d}(g)}(g)$ has a form as required. \square

The next proposition is the second key step in the classification.

Proposition 2.7. *Let $f \in K[[x]]$ and $d = d(f)$. Let $g \in K[[x]]$ be such that $e(f) = e(g)$ and $j^d(f) = j^d(g)$. Then $f \sim_r g$. In particular, if $\mu(f) < \infty$ then f is right d -determined.*

Proof. The proof will be divided into two steps.

Step 1: $e := e(f) = 0$. Since $j^d(f) = j^d(g)$, there exists l with $q + l \geq d + 1$ and $b_{q+l} \neq 0$ s.t.

$$g - f = b_{q+l} x^{q+l} \mod x^{q+l+1},$$

with $d := d(f) = d(g)$, $q := q(f) = q(g)$. We will construct inductively a finite sequence of $f_i, i = 0, \dots, N := \bar{d}(f) - q - l + 1$ with $f_0 = f$ such that $f_i \sim_r f$ for all i and that

$$g - f_i = 0 \mod x^{q+l+i}.$$

If we succeed then by Remark 2.4(c) we have $f \sim_r f_N \sim_r g$. In fact since $q + l \geq d + 1$,

$$l \geq k(f) = \max\left\{ \left\lceil \frac{q-n}{p^{e(n)}-1} \right\rceil \mid m(f) \leq n < q, n \in \text{supp}(f) \right\}.$$

Considering the coordinate change $\varphi_1(x) = x + u_{l+1} x^{l+1}$ with u_{l+1} a solution of the following equation:

$$q c_q X + \sum_{\frac{q-n}{p^{e(n)}-1} = l} (n/p^{e(n)})^{p^{e(n)}} c_n X^{p^{e(n)}} + b_{q+l} = 0$$

and setting $f_1 := \varphi_1(f)$ we get $g - f_1 = 0 \pmod{\mathfrak{m}^{q+l+1}}$ and hence we obtain by induction a finite sequence of $(f_i)_{i=0,\dots,N}$ as required.

Step 2: $e := e(f) > 0$. Taking $\bar{f} \in K[[x]]$ and $\bar{g} \in K[[x]]$ such that $f(x) = \bar{f}(x^{p^e})$, $g(x) = \bar{g}(x^{p^e})$ as in Remark 2.4 we have

$$e(\bar{f}) = e(\bar{g}) = 0, q(\bar{f}) = q(\bar{g}) = q/p^e, \bar{d} := d(\bar{f}) = d(\bar{g}) = d/p^e.$$

Since $j^d(f) = j^d(g)$, $j^{\bar{d}}(\bar{f}) = j^{\bar{d}}(\bar{g})$ and hence $\bar{f} \sim_r \bar{g}$ by the first step. This implies that $f \sim_r g$ with the same coordinate change. \square

Theorem 2.8 (Normal form of univariate power series). *With f , $m(f)$ and $\Lambda(f)$ as above, we have*

$$f \sim_r x^{m(f)} + \sum_{n \in \Lambda(f)} \lambda_n x^n$$

for suitable $\lambda_n \in K$.

Proof. It is sufficient to prove the theorem for the case that $e(f) = 0$ by Remark 2.4. We proceed step by step as in definition of $\Lambda(\Delta)$ with $\Delta := \text{supp}(f)$.

Case 0: $e(m) = 0$ with $m := m(f)$.

The claim follows from Lemma 2.5.

Case 1: $e(m) > 0$ and if

$$e(m) = e(f_0) =: e_0$$

with $\Delta_0 := \{n \in \Delta \mid n < q\}$ and

$$f_0 := \sum_{n \in \Delta_0} c_n x^n.$$

Applying Lemma 2.5 to f_0 we obtain that $f_0 \sim x^m$, i.e. there exists a coordinate change φ such that $\varphi(f_0) = x^m$ and then

$$\varphi(f) = x^m + \varphi(f_1)$$

with

$$f_1 := f - f_0 = \sum_{n \geq q} c_n x^n.$$

We write

$$\varphi(f_1) := \sum_{n \geq q} b_n x^n, \quad b_q \neq 0.$$

Since $k := k(f) = k_\Delta(m)$ we have $m + lp^{e(m)} < q + l$ for all $l < k$. This allows us to eliminate inductively all coefficients b_n in $\varphi(f_1)$ with $e(n) \geq e(m)$ and $q \leq n \leq d$ by a suitable coordinate change (e.g. if n_1 is the minimum for which $e(n_1) \geq e(m)$, $q \leq n_1 \leq d$ and $b_{n_1} \neq 0$, then the coordinate change

$$\varphi_1(x) = x + u_{l+1}x^{l+1}$$

with $l = \frac{n_1 - m}{p^{e(m)}}$ and u_{l+1} a solution of the equation:

$$(m/p^{e(m)})^{p^{e(m)}} X^{p^{e(m)}} + b_{n_1} = 0,$$

makes the coefficient of x^{n_1} vanishing, and the terms of exponents less than n_1 do not change). This completes the proof.

Case 2: $e(m) > e_0 > 1$.

Then applying Lemma 2.6 to f_0 we get

$$f_0 \sim_r x^m + \sum_{n \in \Lambda(f_0)} \lambda_n x^n$$

for suitable λ_n , i.e.

$$\varphi(f_0) = x^m + \sum_{n \in \Lambda(f_0)} \lambda_n x^n$$

for some coordinate change φ . Then

$$\varphi(f_0) = x^m + \sum_{n \in \Lambda_0(\Delta)} \lambda_n x^n \pmod{x^q}$$

and hence

$$\varphi(f) \sim_r x^m + \sum_{n \in \Lambda_0(\Delta)} \lambda_n x^n + \sum_{n \in \Lambda_1(\Delta)} \lambda_n x^n$$

by Proposition 2.7. This proves the claim.

Case 3: $e(m) > e_0 = 1$ and if $k(f) > k_\Delta(q_0)$ with

$$q_0 := q(\Delta_0) = \min\{n \in \Delta_0 \mid e(n) = e_0\}.$$

Using the same argument as in Case 2 gives us the claim.

Case 4,5: If $e(m) > e_0 = 1$ and $k(\Delta) = k_\Delta(q_0)$.

This is done by the same method as in Case 1 and Case 2. □

3. RIGHT MODALITY

Theorem 3.1. *Let $\text{char} K = p > 0$. Let $f \in \langle x \rangle \subset K[[x]]$ be a univariate power series such that its Milnor number $\mu := \mu(f)$ is finite. Then*

$$\mathcal{R}\text{-mod}(f) = \lfloor \mu/p \rfloor.$$

For the proof we need the following lemma which is proven in [GN12, Prop. 2.16] for unfoldings but the proof works in general (for algebraic families of power series).

Let us introduce the notion of unfoldings. Let T be an affine variety over K with the structure sheaf \mathcal{O} and its algebra of global section $\mathcal{O}(T)$. An element $f_{\mathbf{t}}(x) := F(x, \mathbf{t}) \in \mathcal{O}(T)[[x]]$ is called an *algebraic family of power series* over T . A family $F(x, \mathbf{t}) \in \mathcal{O}(T)[[x]]$ is called an *unfolding* of f at $\mathbf{t}_0 \in T$ over T if $f_{\mathbf{t}_0} = f$ and $f_{\mathbf{t}} \in \langle x \rangle$ for all $\mathbf{t} \in T$.

Remark 3.2. Let $f \in \langle x \rangle \subset K[[x]]$ with the Milnor number $\mu < \infty$. Then the system $\{x, x^2, \dots, x^\mu\}$ is a basis of the algebra $\langle x \rangle / \langle x \cdot \frac{\partial f}{\partial x} \rangle$. By [GN12, Prop. 2.25] the unfolding over \mathbb{A}^μ ,

$$f_{\mathbf{t}}(x) := f + \sum_{i=1}^{\mu} t_i \cdot x^i$$

is an algebraic representative of the semiuniversal deformation with trivial section of f .

Lemma 3.3. *With f and $f_{\mathbf{t}}(x)$ as above, assume that there exist a finite number of algebraic families of power series $h_{\mathbf{t}}^{(i)}(\mathbf{x})$, $i \in I$ over varieties $T^{(i)}$, $i \in I$ and an open subset $U \subset \mathbb{A}^\mu$ satisfying: for all $\mathbf{t} \in U$ there exists an $i \in I$ and $\mathbf{t}_i \in T^{(i)}$ such that $f_{\mathbf{t}}(\mathbf{x})$ is right equivalent to $h_{\mathbf{t}_i}^{(i)}(\mathbf{x})$. Then*

$$\mathcal{R}\text{-mod}(f) \leq \max_{i=1, \dots, l} \dim T^{(i)}.$$

Proof of Theorem 3.1. We first prove the inequality $\mathcal{R}\text{-mod}(f) \leq \lfloor \mu/p \rfloor$. Indeed, let $k = 2\mu(f)$ and let

$$I := \{\Delta \subset \mathbb{N} \mid q(f) \in \Delta, 1 \leq n \leq k \ \forall n \in \Delta\}.$$

With the unfolding $f_{\mathbf{t}}(x)$ be as above, by the upper semicontinuity of the Milnor number (cf. [GN12]), there exists an open subset U of $\mathbf{t}_0 \in \mathbb{A}^\mu$ such that $\mu(f_{\mathbf{t}}) \geq \mu(f)$ for all $\mathbf{t} \in U$. This implies that $\text{supp}(f_{\mathbf{t}}) \in I$ for all $\mathbf{t} \in U$. We can easily verify that the finite set of families

$$h_{\mathbf{s}_\Delta}(x) := x^{m(\Delta)} + \sum_{n \in \Lambda(\Delta)} s_\Delta^{(n)} x^n, \quad \Delta \in I$$

over $A_\Delta \equiv \mathbb{A}^{l_\Delta}$ with $l_\Delta = \#\Lambda(\Delta)$, satisfies the assumption of Lemma 3.3 and hence

$$\mathcal{R}\text{-mod}(f) \leq \max_{\Delta \in I} \#\Lambda(\Delta) \leq \lfloor q/p \rfloor = \lfloor \mu/p \rfloor$$

by proposition 2.3.

In order to prove the other inequality we consider the two following cases.

Case 1: $m(f) = p$.

Then $q := q(f) = \mu(f) + 1$,

$$\Lambda(f) = \{n \geq q \mid e(n) = 0\}$$

and $\#\Lambda(f) = \lfloor q/p \rfloor$ due to Proposition 2.3. It follows from Proposition 2.8 that

$$f \sim_r f_\lambda := x^p + \sum_{n \in \Lambda(f)} \lambda_n x^n$$

for suitable $\lambda_n \in K$. Let us show that if $f_\lambda \sim_r f_{\lambda'}$ then $(\lambda_n)_{n \in \Lambda(f)} = (\lambda'_n)_{n \in \Lambda(f)}$. In fact, since $f_\lambda \sim_r f_{\lambda'}$, there exists a coordinate change

$$\varphi := ax + a_l x^{l+1} + \dots$$

such that

$$\varphi(f_\lambda) = f_{\lambda'}.$$

Then $a^p = 1$ and therefore $a = 1$. If $l < k(f)$ then the point $p(l+1) \in \text{supp}(\varphi(f_\lambda))$ but $p(l+1) \notin \text{supp}(f_{\lambda'})$, that is $\varphi(f_\lambda) \neq f_{\lambda'}$. The contradiction gives us $l \geq k(f)$. It then follows from elementary calculations that $j^d(f_\lambda) = j^d(\varphi(f_\lambda)) = j^d(f_{\lambda'})$ and hence $(\lambda_n)_{n \in \Lambda(f)} = (\lambda'_n)_{n \in \Lambda(f)}$. This implies that $\mathcal{R}\text{-mod}(f) \geq \#\Lambda(f) = \lfloor q/p \rfloor = \lfloor \mu/p \rfloor$.

Case 2: $m(f) > p$.

By the upper semicontinuity of the right modality (cf. [GN12]) one has $\mathcal{R}\text{-mod}(f) \geq \mathcal{R}\text{-mod}(f_t)$ with $f_t = f + t \cdot x^p$ for all t in some neighbourhood W of 0 in \mathbb{A}^1 . Take a $t_0 \in W \setminus \{0\}$ then $\mathcal{R}\text{-mod}(f_{t_0}) = \lfloor \mu/p \rfloor$ by the first step and hence

$$\mathcal{R}\text{-mod}(f) \geq \mathcal{R}\text{-mod}(f_{t_0}) = \lfloor \mu/p \rfloor.$$

□

Remark 3.4. We have $\mathcal{R}\text{-mod}(f) \geq \sharp\Lambda(f)$ by Theorem 3.1 and Proposition 2.3 with equality if $m(f) \leq p$. Moreover, if $m(f) = p$, then $f_\lambda \sim_r f_{\lambda'}$ for $\lambda, \lambda' \in \Lambda(f)$ implies $\lambda = \lambda'$, which follows from the proof of Theorem 3.1.

The example $f = x^{p+1}$ with $\mathcal{R}\text{-mod}(f) = 1$ but $\Lambda(f) = \emptyset$ shows that a strict inequality $\mathcal{R}\text{-mod}(f) > \sharp\Lambda(f)$ can happen.

With f and the semiuniversal unfolding $f_t(x)$ as in Remark 3.2 we define

$$\Sigma_\mu := \{\mathbf{t} \in \mathbb{A}^\mu \mid \mu(f_{\mathbf{t}}) = \mu\}$$

the μ -constant stratum of the unfolding f_t .

Corollary 3.5. *Let $f \in \langle x \rangle \subset K[[x]]$ with the Milnor number $\mu < \infty$. Then*

$$\mathcal{R}\text{-mod}(f) = \dim \Sigma_\mu.$$

Proof. For each $\mathbf{t} = (t_1, \dots, t_\mu) \in \mathbb{A}^\mu$, if the set $N_{\mathbf{t}} := \{i = 1, \dots, \mu \mid t_i \neq 0, e(i) = 0\}$ is not empty, then $\mu(f_{\mathbf{t}}) = n - 1 < \mu$ with $n := \min\{i \mid i \in N_{\mathbf{t}}\}$. This implies that

$$\Sigma_\mu = \{\mathbf{t} = (t_1, \dots, t_\mu) \in \mathbb{A}^\mu \mid t_i = 0, \forall e(i) = 0\}.$$

It yields $\dim \Sigma_\mu = \lfloor \frac{\mu}{p} \rfloor$ and hence $\mathcal{R}\text{-mod}(f) = \dim \Sigma_\mu$ by Theorem 3.1. □

REFERENCES

- [Arn72] Arnol'd V. I., *Normal forms for functions near degenerate critical points, the Weyl groups of A_k, D_k, E_k and Lagrangian singularities*, Functional Anal. Appl. 6 (1972) 254-272.
- [BGM12] Boubakri Y., Greuel G.-M., and Markwig T., *Invariants of hypersurface singularities in positive characteristic*, Rev. Mat. Complut. 25 (2012), 61-85.
- [GN12] Greuel G.-M., Nguyen H. D., *Right simple singularities in positive characteristic*, arXiv:1206.3742 (2012).
- [Ng12] Nguyen H. D., *Classification of singularities in positive characteristic*, Ph.D. thesis, TU Kaiserslautern, 2012.

NGUYEN HONG DUC

INSTITUTE OF MATHEMATICS, 18 HOANG QUOC VIET ROAD, CAU GIAY DISTRICT
10307, HANOI.

E-mail address: nhduc@math.ac.vn

UNIVERSITÄT KAISERSLAUTERN, FACHBEREICH MATHEMATIK, ERWIN-SCHRÖDINGER-STRASSE,
67663 KAISERSLAUTERN

E-mail address: dnguyen@mathematik.uni-kl.de